Exercise 1 (Young's Inequality)**.** Let $1 \leq p, q, r \leq \infty$ be such that $\dfrac{1}{p} + \dfrac{1}{q} = 1 + \frac{1}{r}$. Prove that for all $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$, for $\mathscr{L}^d$ almost every $x \in \mathbb{R}^d$, the integral

$$f * g(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy$$

converges, and that we have

$$\|f * g\|_{\mathrm{L}^r(\mathbb{R}^d)} \leq \|f\|_{\mathrm{L}^p(\mathbb{R}^d)} \|g\|_{\mathrm{L}^q(\mathbb{R}^d)}$$

Hint: First treat the case $p = q = 1$, then $p = 1$ and $1 < q \leq \infty$. In general, write $|f(x-y)||g(y)| = (|f(x-y)|^{\alpha}|g(y)|^{\beta})(|f(x-y)|^{1-\alpha}|g(y)|^{1-\beta})$ for well-chosen $0 < \alpha, \beta < 1$ and apply Hölder's inequality.

Exercise 2 (A Direct Proof of the Poincaré Inequality)**.** Let $B = B(x_0, r) \subset \mathbb{R}^d$ be an open ball and let $u \in C^1(B)$ be a function of zero average on B, i.e., that satisfies

$$\int_B u(x)dx = 0.$$

1. Show that

$$\frac{1}{|B|}\int_B u^2(x)dx = \frac{1}{2|B|^2}\int_{B\times B}(u(x)-u(y))^2 dx\,dy.$$

2. Deduce that

$$\frac{1}{|B|}\int_B u^2 dx \leq \frac{(2r)^2}{2|B|^2}\int_{B\times B}\left(\int_0^1 |\nabla u(tx+(1-t)y)|^2 dt\right)dx\,dy.$$

Show that for all $z \in B$, we have

$$\int_B \mathbf{1}_{tB+(1-t)y}(z)dy \leq \min\left\{1, \frac{t^d}{(1-t)^d}\right\}|B|.$$

3. Deduce that there exists a constant $C < \infty$ (independent of u, x_0, and r) such that

$$\left(\frac{1}{|B|}\int_B u^2 dx\right)^{\frac{1}{2}} \leq C\,r\left(\frac{1}{|B|}\int_B |\nabla u|^2 dx\right)^{\frac{1}{2}}.$$

4. Conclude that for all $u \in W^{1,2}(B)$, we have

$$\left(\frac{1}{|B|}\int_B |u-\overline{u}_B|^2 dx\right)^{\frac{1}{2}} \leq C\,r\left(\frac{1}{|B|}\int_B |\nabla u|^2 dx\right)^{\frac{1}{2}},$$

where $\overline{u}_B$ is the average of u on B.

Exercise 3. Show that for all $0 < s < 1$, we have

$$H^s(\mathbb{R}^d) = L^2(\mathbb{R}^d) \cap \left\{u : \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}dx\,dy < \infty\right\}.$$

Exercise 4. Let $d \geq 2$ and $s > \dfrac{d}{2}$.

1. Show that there exists $C_1 < \infty$ such that for all $1 \leq j, k \leq d$ and for all $u \in H^2(\mathbb{R}^d)$, we have

$$\left\| \frac{\partial^2}{\partial x_j \partial x_k} u \right\|_{\mathrm{L}^2(\mathbb{R}^d)} \leq C_1 \left\| \Delta u \right\|_{\mathrm{L}^2(\mathbb{R}^d)}.$$

2. Show that there exists $C_2 < \infty$ such that for all $u \in H^2(\mathbb{R}^3)$, we have

$$\|u\|_{\mathrm{L}^\infty(\mathbb{R}^3)} \leq C_2 \left(\|\nabla u\|_{\mathrm{L}^2(\mathbb{R}^d)} + \|\Delta u\|_{\mathrm{L}^2(\mathbb{R}^d)} \right).$$

   By a scaling argument, deduce the following inequality (for some $C_3 < \infty$ independent of u)

$$\|u\|_{\mathrm{L}^\infty(\mathbb{R}^3)} \leq C_3 \|\nabla u\|_{\mathrm{L}^2(\mathbb{R}^3)}^{\frac{1}{2}} \|\Delta u\|_{\mathrm{L}^2(\mathbb{R}^3)}^{\frac{1}{2}}.$$